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RADICALS OF CROSSED PRODUCTS OF ENVELOPING ALGEBRAS

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ABSTRACT

Let L be a Lie algebra over a field K which acts as K-derivations on a K-algebra R. Then this action determines a crossed product $R * U(L)$ where $U(L)$ is the enveloping algebra of L. The goal of this paper is to describe the Jacobson radical of $R * U(L)$ for $L \neq 0$. We are most successful when R is a p.i. algebra or Noetherian. In more general situations we at least obtain upper and lower bounds for $J(R * U(L))$ which are ideals extended from R. Furthermore, we offer an interesting example in all characteristics of a commutative K-algebra C which admits a derivation δ such that C is δ-prime but not semiprime.

Let L be a Lie algebra over the commutative ring K, such that L is a free K-module, and let $U(L)$ denote its universal enveloping algebra. If R is a K-algebra and L acts on R as K-derivations, then this action determines in a natural manner a ring generated by R and $U(L)$. This K-algebra is denoted by $R * U(L)$ and is called the crossed product of R by $U(L)$. The aim of this paper is to describe the Jacobson radical $J(R * U(L))$ when $L \neq 0$. Our main result, Corollary 3.5, asserts that if R is a p.i. algebra, then $J(R * U(L)) = N * U(L)$ where N is the largest L-invariant nil ideal of R; moreover $J(R * U(L))$ is nil in this case. By a somewhat easier argument, we obtain the same conclusion for R any Noetherian algebra.

For an arbitrary algebra R, we find upper and lower bounds for $J(R * U(L))$

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which are ideals extended from R. For example, if K has characteristic 0, we prove that

$$N * U(L) \subseteq J(R * U(L)) \subseteq M * U(L)$$

where N is the prime radical of R and M is the largest L-stable ideal contained in $J(R)$ (Corollaries 2.4 and 3.10). Furthermore M can be replaced by the nil radical of R if some $0 \neq x \in L$ acts as an inner derivation on R.

Previous results on this problem were of two kinds. The first, when L acts trivially, include Amitsur's well-known theorem [2] on polynomial rings $R[x]$ and Irving's theorem [10] on enveloping algebras $U(L)$ for L a Lie algebra which is also a finite K-module (here $R = K$). We remark that the latter theorem now has a very short proof [3]. For the second kind, when the action of L is non-trivial, much less is known. In fact only the rank one case, where $L = Kx$ and $R * U(L) = R[x; \delta]$ is an Ore extension, has been studied. In this case, the radical was determined for R a commutative ring in [7] and for R Noetherian in [11]. By iteration, the result of [11] extends to solvable Lie algebras L.

Thus our results generalize those of [7, 10, 11] to actions of arbitrary Lie algebras.

In addition, we study a rather interesting example, namely the countable-dimensional exterior algebra E over an arbitrary field K. We show (Proposition 1.3) that E can be given a derivation δ such that E is δ-prime. Since E is not semiprime, this therefore answers a question of [8]. By taking the center of E, one even obtains a commutative example. It follows that $S = E[y; \delta]$ is prime and, since E is a p.i. algebra, $J(S) = N[y; \delta]$ is a nil ideal. Thus $J(S)$ gives a natural example of a prime nil ring which is generated by two elements. Moreover, E is a Jacobson ring, whereas S is not (Proposition 2.3). The latter example in characteristic 0 is new; an example in characteristic $p \neq 0$ is given in [6]. Finally E and S have interesting growth properties. Although E is locally finite, S does not have polynomial growth even though it is generated by two elements.

We now describe what we mean by a crossed product, following [12, Chapter 1, Section 7]. First note that since L is free over K, the Poincaré–Birkhoff–Witt theorem holds, and thus the standard monomials in a K-basis $\{x_i\}$ for L form a basis for $U(L)$. A K-algebra S containing R is called a *crossed product* of R by $U(L)$, and writen $R * U(L)$, provided there is a K-module embedding of L into S, $x \to \bar{x}$, such that for all $x, y \in L$, $r \in R$,

(i) $\bar{x}r - r\bar{x} = \delta_x(r) \in R$, where $\delta_x \in \mathrm{Der}_K(R)$.

(ii) $\bar{x}\bar{y} - \bar{y}\bar{x} = \overline{[x, y]} + t(x, y)$, where $t : L \times L \to R$.

(iii) S is a free right (and left) R-module with the standard monomials in $\{\bar{x}_i\}$ as basis.

Although crossed products are defined in a somewhat different manner in [5], the two notions are equivalent. Note that if the cocyle $t \equiv 0$, then $R * U(L)$ becomes the more familiar skew enveloping algebra, or differential polynomial ring, written $R \# U(L)$.

In fact, the results of Section 2 are valid for more general algebras than crossed products; we thank A. Joseph for pointing this out to us. A K-algebra S is called a *Lie extension* of R by L, and written $R\langle L\rangle$, provided S is generated by a subalgebra R and a subspace L such that for all $x, y \in L$, $r \in R$,

(i) $xr - rx = \delta_x(r) \in R$, where $\delta_x \in \mathrm{Der}_K(R)$,

(ii) $xy - yx \in L + R$.

Note that L is not assumed to be a Lie algebra here. In particular a homomorphic image of a crossed product $R * U(L)$ is a Lie extension.

§1. L-prime rings

In this section we briefly discuss L actions on rings. We are especially concerned with the ideals which are L-stable. To be precise, let L be a vector space over K, let R be a K-algebra and let $\delta : L \to \mathrm{Der}_K R$ be any map from L into the algebra of K-derivations of R. Thus each $x \in L$ determines a derivation δ_x of R. Note that we do not assume here that L is a Lie algebra or that δ is a Lie homomorphism. An ideal I of R is said to be L-stable or L-invariant if $\delta_x(I) \subseteq I$ for all $x \in L$. Obviously this can be checked one derivation at a time. The following result is known; however our proofs of (i) and (ii) seem easier than others in the literature.

LEMMA 1.1. *Let* char $K = 0$. *Then L stabilizes*

(i) *the nil radical of R,*

(ii) *the sum of all nilpotent ideals of R,*

(iii) *the prime radical of R,*

(iv) *each minimal prime of R.*

PROOF. Let $\delta \in \mathrm{Der}_K(R)$ and let $A \lhd R$. Then $A + \delta(A) \lhd R$ since $r\delta(a) \equiv \delta(ra) \bmod A$ and $\delta(a)r \equiv \delta(ar) \bmod A$ for all $r \in R$, $a \in A$.

(i) For this we need only show that if $A \lhd R$ is nil, then so is $A + \delta(A)$. In fact we need only check that $\delta(A)$ is nil modulo A. Let $a \in A$ and say $a^n = 0$. Then we have easily

$$0 = \delta^n(a^n) \equiv n!\,(\delta a)^n \bmod A.$$

Since char $K = 0$ we conclude that $(\delta a)^n \in A$ as required.

(ii) Here we must show that if $A \triangleleft R$ is nilpotent, then $\delta(A)$ is nilpotent modulo A and the same argument works. Indeed if $A^n = 0$, then

$$0 = \delta^n(A^n) \equiv n!\,(\delta A)^n \bmod A,$$

so $(\delta A)^n \subseteq A$.

(iv) This is proved in [8, Proposition 1.1] and (iii) is immediate from either (ii) or (iv).

The standard counterexample in characteristic $p > 0$ is as follows. Let char $K = p$ and set $R = K[x]/(x^p)$. Then $\delta = \partial/\partial x$ is a derivation of R and all four of the ideals discussed in Lemma 1.1 are equal to xR which is not δ-stable. In fact, R is a δ-simple ring, that is it has no nontrivial δ-stable ideals.

In view of the subject of this paper, it is natural to ask whether JR is necessarily L-stable. The answer is "no" in any characteristic. Indeed let $R = K[x]_x$ be the set of all rational functions $f(x)/g(x)$ with $g(0) \neq 0$. Then $\delta = \partial/\partial x$ is a derivation of R but $JR = xR$ is not δ-stable.

Again let L act on R. We say that R is L-prime if for all L-stable ideals $0 \neq A, B$ we have $AB \neq 0$. It is clear that if R is prime then it is L-prime. For the converse we know at least

LEMMA 1.2. *Let L act on R.*

(i) *If R is semiprime then every annihilator ideal is L-stable.*

(ii) *If R is semiprime and L-prime, then it is prime.*

(iii) *Assume R has a unique largest nilpotent ideal and that* char $K = 0$. *If R is L-prime then it is prime.*

PROOF. (i) Say $A \triangleleft R$ and $B = \mathbf{r}_R(A)$. Since $\delta(A^2) \subseteq A$ and $AB = 0$ we have

$$0 = \delta(A^2B) = A^2\delta(B) \supseteq (A\delta(B))^2$$

so $\delta(B) \subseteq \mathbf{r}_R(A) = B$.

(ii) Let $A, B \triangleleft R$ with $AB = 0$. We may assume first that $B = \mathbf{r}_R(A)$ and then that $A = \mathbf{l}_R(B)$. But then A and B are both L-stable so, since R is L-prime, one of A or B must be zero.

(iii) If N is the unique largest nilpotent ideal of R then, by Lemma 1.1(ii), N is L-stable. But R is L-prime and $N^n = 0$ so $N = 0$. Therefore R is semiprime and hence prime by (ii).

Part (iii) of course always applies when R is right Noetherian. We note however that (iii) is false in characteristic $p > 0$. The ring $R = K[x]/(x^p)$ with $\delta = \partial/\partial x$ is an appropriate counterexample. This also shows that the semiprime hypothesis is required in (ii). An example for this valid in all characteristics, including characteristic zero, is as follows. This answers a question in [8].

Proposition 1.3. *Let K be a field and let E be the Grassmann (exterior) algebra over K generated by the countably many elements $x_1, x_2, x_3, \ldots$. Then there exists a K-derivation δ of E with $\delta(x_i) = x_{i+1}$ for all i. Furthermore for this δ, E and its center $\mathbf{Z}(E)$ are both δ-prime rings which are not semiprime.*

Proof. We first observe that δ defines a derivation on E. To this end let $F = K\langle X_1, X_2, X_3, \ldots\rangle$ be a free K-algebra and map F onto E via $X_i \to x_i$. Then by definition of E, the kernel of this map is the ideal generated by X_i^2 and $X_iX_j + X_jX_i$ for all i, j. Now $\delta(X_i) = X_{i+1}$ certainly extends to a derivation on F and thus δ will yield a derivation of E provided that the ideal of relations is δ-stable. But

$$\delta(X_i^2) = X_iX_{i+1} + X_{i+1}X_i$$

and

$$\delta(X_iX_j + X_jX_i) = (X_{i+1}X_j + X_jX_{i+1}) + (X_iX_{j+1} + X_{j+1}X_i)$$

so this fact is clear.

We can now proceed as in [8, Example 1.6]. For convenience, if $i_1 < i_2 < \cdots < i_n$ we write $x_{i_1}x_{i_2}\cdots x_{i_n} = x_S$ where S is the set $\{i_1, i_2, \ldots, i_n\}$. Then the collection of these monomials x_S yields a K-basis for E. We call a monomial of the form $x_ix_{i+1}x_{i+2}\cdots x_{i+m}$ a consecutive segment starting at i. We note that any nonzero ideal I of E contains a consecutive segment starting at $i = 1$. Indeed let $0 \neq \alpha = \sum k_Sx_S \in I$ and let $U = \{1, 2, \ldots, m\}$ be chosen with $U \supseteq S$ for all x_S in the support of α. Furthermore let T be of minimal size with $k_T \neq 0$. Notice that if x_S is in the support of α and $x_S \cdot x_{U\setminus T} \neq 0$, then S must be disjoint from $U \setminus T$ so $S \subseteq T$ and hence $S = T$. Thus $k_Tx_U = \pm\, \alpha \cdot x_{U\setminus T} \in I$.

Next we observe that if $s \geqq 2n$ then

$$(*) \quad \delta^n(x_ix_{i+1}\cdots x_{i+n-1})\cdot x_{i+n+1}x_{i+n+2}\cdots x_{i+s-1} = x_{i+1}x_{i+2}\cdots x_{i+s-1}.$$

For this we may assume for convenience that $i = 1$ and proceed by induction on n. We wish to consider

$$\beta = \delta^n(x_1x_2\cdots x_n)\cdot x_{n+2}x_{n+3}\cdots x_s.$$

Since $\delta(x_1x_2\cdots x_n) = x_1x_2\cdots x_{n-1}\cdot x_{n+1}$ we have

$$\beta = \delta^{n-1}(x_1x_2\cdots x_{n-1}\cdot x_{n+1})\cdot x_{n+2}x_{n+3}\cdots x_s.$$

But $s \geqq 2n$ implies that any further derivative of x_{n+1} will annihilate $x_{n+2}x_{n+3}\cdots x_s$. Thus in computing β we may essentially view x_{n+1} as a constant. This yields

$$\beta = \delta^{n-1}(x_1x_2\cdots x_{n-1})\cdot x_{n+1}x_{n+2}\cdots x_s$$

and the fact follows by induction on n.

As a consequence of the two previous paragraphs we see that any nonzero δ-stable ideal I of E contains a consecutive segment starting at i for all $i \geqq 1$. In fact the $i = 1$ case has already been proved and formula (*) shows that i implies $i + 1$.

It is now a simple matter to prove the result. If A and B are nonzero δ-stable ideals of E, then A contains $x_1x_2\cdots x_{n-1}$ for some $n \geqq 2$ and B contains $x_nx_{n-1}\cdots x_t$ for some $t \geqq n$. Thus $AB \neq 0$ and E is δ-prime. Hence so is $\mathbf{Z}(E)$ since δ acts on $\mathbf{Z}(E)$ and since any δ-stable ideal of $\mathbf{Z}(E)$ extends to one of E. Finally x_1x_2 is a central element of square zero so neither E nor $\mathbf{Z}(E)$ is semiprime.

In the preceding example, every annihilator ideal is nilpotent. In fact, we do not know of an L-prime ring which does not have this property.

§2. The lower bound

We now formally begin our work on $J(R * U(L))$. In this section we obtain a lower bound for this ideal, in the more general setting of Lie extensions $R\langle L\rangle$. Indeed we show that if N is an L-stable ideal of R which is generated by nilpotent ideals, then the extended ideal $N\langle L\rangle$ is nil and hence contained in the Jacobson radical.

We fix some notation. Let $S = R\langle L\rangle$ be a Lie extension of R by L over K. If X is a K-subspace of L, then for convenience we write

$$X^{(n)} = K + X + X^2 + \cdots + X^n = (K + X)^n.$$

Furthermore if $(a_1, a_2 \dots, a_t)$ is a t-tuple of nonnegative integers, then we denote the tuple by its corresponding letter a and we write $|a| = a_1 + a_2 + \cdots + a_t$. Recall that if $x \in L$ then $xr = rx + \delta_x(r)$ for all $r \in R$.

LEMMA 2.1. *Let X be a K-subspace of L and let V be a K-subspace of R. Define V_n inductively by $V_0 = V$ and*

$$V_{n+1} = V_n + \sum_{x \in X} \delta_x(V_n).$$

Then for all $m, t \geqq 1$

$$(VX^{(m)})^t \subseteq \sum_{|a| = mt} SV_{a_1}V_{a_2} \cdots V_{a_t}.$$

PROOF. First observe that for any $x \in X$

$$\delta_x(V_{a_1}V_{a_2} \cdots V_{a_s}) \subseteq \sum_{|b| = 1} V_{a_1+b_1}V_{a_2+b_2} \cdots V_{a_s+b_s}.$$

Hence since $xr = rx + \delta_x(r)$ for all $r \in R$ we have

$$V_{a_1}V_{a_2} \cdots V_{a_s}X \subseteq \sum_{|b| = 1} SV_{a_1+b_1}V_{a_2+b_2} \cdots V_{a_s+b_s}.$$

It now follows by induction on $m \geqq 1$ that

$$(**) \qquad V_{a_1}V_{a_2} \cdots V_{a_s}X^{(m)} \subseteq \sum_{|b| = m} SV_{a_1+b_1}V_{a_2+b_2} \cdots V_{a_s+b_s}.$$

Finally we prove the lemma by induction on t. The case $t = 1$ follows from (**) with $s = 1$ and $a_1 = 0$. If the inclusion holds for $t - 1$ then

$$(VX^{(m)})^t \subseteq \sum_{|a| = m(t-1)} SV_{a_1}V_{a_2} \cdots V_{a_{t-1}}(V_0X^{(m)}).$$

Another application of (**) with $s = t$ yields the result.

The next theorem extends one direction of [7, Theorem 3.3] from the case of $L = Kx$ to arbitrary L. Given our Lemma 2.1, essentially the same proof in [7] can be used.

THEOREM 2.2. *Let N be an L-stable ideal of R generated by nilpotent ideals. Then $N\langle L \rangle$ is a nil ideal of $R\langle L \rangle$.*

PROOF. Since N is L-stable we know that $N\langle L \rangle \lhd S = R\langle L \rangle$. Let $\alpha \in N\langle L \rangle$. Then there exists a finite dimensional subspace V of N and a finite dimensional subspace X of L with $\alpha \in VX^{(m)}$ for some $m \geqq 1$. It suffices to prove that $VX^{(m)}$ is nilpotent and we use the notation of the preceding lemma. It is clear that V_{2m-1} is finite dimensional and that $V_{2m-1} \subseteq N$ since N is L-stable. Thus since N is generated by nilpotent ideals there exists $I \lhd R$ with $V_{2m-1} \subseteq I$ and $I^s = 0$.

Set $t = 2s$. We claim that $V_{a_1} V_{a_2} \cdots V_{a_t} = 0$ if $|a| = mt$. Indeed, since $mt = |a| = a_1 + a_2 + \cdots + a_t$, there are at most $s = t/2$ of the a_i's which are $\geqq 2m$. Thus the remaining V_{a_i} are contained in I and hence

$$V_{a_1} V_{a_2} \cdots V_{a_t} \subseteq I^s = 0.$$

Lemma 2.1 now implies that $(VX^{(m)})^t = 0$.

This result has numerous consequences. We start with an example in differential polynomial rings analogous to the skew polynomial example in [13]. A characteristic $p > 0$ example is contained in [6].

PROPOSITION 2.3. *Let E be the Grassmann algebra over K generated by the countably many elements $x_1, x_2, x_3, \ldots$ and let δ be the derivation of E defined by $\delta(x_i) = x_{i+1}$. Then E is a Jacobson ring but the Ore extension $S = E[y; \delta]$ is not. Indeed S is prime but $J(S) = N[y; \delta] \neq 0$ where N is the prime radical of E.*

PROOF. Since $E/N = K$, N is the unique prime of E and E is clearly Jacobson. We know by Proposition 1.3 that δ defines a derivation on E and that E is δ-prime. Thus, by [5, Theorem 2.6], $S = E[y; \delta]$ is prime. Furthermore since N is δ-stable and generated by nilpotent ideals, Theorem 2.2 implies that $N[y; \delta] \subseteq J(S)$. Finally

$$S/N[y; \delta] \simeq (E/N)[y; \delta] \simeq K[y]$$

is semiprimitive so we conclude that $N[y; \delta] = J(S)$.

Note that $J(S)$ is a prime nilring generated by x_1 and y.

We remark that the K-algebra $S = E[y; \delta]$ above has rather interesting growth properties. Since the argument is essentially a modification of the work of [14], we just sketch it here and refer the reader to that paper for complete details. Note first that S is a finitely generated K-algebra with generating vector space $V = K + Kx_1 + Ky$. It follows easily that V^n has a k-basis consisting of all monomials of the form $x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k} y^b$ with $a_i = 0$ or 1, $b \geqq 0$ and $\Sigma_i \, ia_i + b \leqq n$. By adding an additional parameter c, we see that $d_n = \dim_K V^n$ is equal to the number of infinite tuples $(a_1, a_2, \ldots, a_k, \ldots, b, c)$ of integers with $a_i = 0$

or 1, $b \geqq 0$, $c \geqq 0$ and $\Sigma_i ia_i + b + c = n$. Thus the generating function for these dimensions is given by

$$\sum_{n=0}^{\infty} d_n \zeta^n = \prod_{i=1}^{\infty} (1 + \zeta^i) \cdot (1 + \zeta + \zeta^2 + \cdots)^2$$

$$= (1 - \zeta)^{-2} \cdot \prod_{i=1}^{\infty} (1 + \zeta^i).$$

To obtain a lower bound for d_n, we note that the sum of the positive integers $< \sqrt{n}$ is at most n. Thus $\Sigma_{i=1}^{\sqrt{n}} ia_i \leqq n$ for all choices of $a_i = 0$ or 1 and hence $d_n \geqq 2^{\sqrt{n}-1}$. This shows that S does not have polynomial growth so $\operatorname{GKdim} S = \infty$ even though $\operatorname{GKdim} E = 0$ since E is locally finite dimensional. In the other direction we again fix n and note that $a_i \neq 0$ implies that $i \leqq n$. Furthermore we have

$$n \geqq \sum_{i=\sqrt{n}}^{n} ia_i \geqq \sqrt{n} \sum_{i=\sqrt{n}}^{n} a_i$$

so at most $\sqrt{n}$ of the a_i in the range $\sqrt{n} \leqq i \leqq n$ can be nonzero. It therefore follows that at most $2\sqrt{n}$ of the a_i in the range $1 \leqq i \leqq n$ can be nonzero and this yields a crude upper bound of $n^{2\sqrt{n}}$ for the number of choices for $(a_1, a_2, \ldots)$. Taking into account the b and c terms then yields $d_n \leqq (n+1)^2 \cdot n^{2\sqrt{n}}$ and we conclude easily that S does not have exponential growth.

Corollary 2.4. *Let R be an algebra over a field K of characteristic 0 and let N be the prime radical of R. If $R\langle L\rangle$ is a Lie extension then $N\langle L\rangle$ is a nil ideal of $R\langle L\rangle$.*

Proof. We use the characteristic 0 assumption twice. First it implies, by Lemma 1.1(iii), that N is an L-invariant ideal of R and hence that $N\langle L\rangle$ is an ideal of $S = R\langle L\rangle$. Now choose $M \lhd R$ maximal subject to $M \subseteq N$, M is L-stable and $M\langle L\rangle$ is nil. The goal is to show that $M = N$. Since the extension of a nil ideal by a nil ideal is nil, we may mod out by $M\langle L\rangle$ and assume that $M = 0$.

If $N \neq 0$, then N contains nonzero nilpotent ideals of R and we let N_0 be the sum of these ideals. Since $\operatorname{char} K = 0$, it follows from Lemma 1.1(ii) that $N_0 \neq 0$ is also L-stable. Theorem 2.2 now implies that $N_0\langle L\rangle$ is a nil ideal of $R\langle L\rangle$ contradicting the maximality of $M = 0$. Thus $N = 0 = M$ as required.

It is natural to try to salvage some of this in characteristic $p > 0$. Of course the prime radical of R need not be L-invariant, but it does contain a largest L-invariant ideal, say N. The question then is whether $N\langle L\rangle$ is nil. We have

COROLLARY 2.5. *Let $R\langle L\rangle$ be given and let N be the largest L-invariant nil ideal of R. If R is either right Noetherian or a p.i. algebra, then $N\langle L\rangle$ is nil.*

PROOF. If R is Noetherian, then N is nilpotent so the result is obvious. We assume that R satisfies a polynomial identity of degree d. As in the proof of Corollary 2.4, we may assume that the largest L-invariant ideal M of R with $M \subseteq N$ and $M\langle L\rangle$ nil is $M = 0$. The goal is to show that $N = 0$.

By [1], $N_0 = N^{[d/2]}$ is generated by nilpotent ideals of R and it is surely L-stable since it is a power of N. Thus by Theorem 2.2, $N_0\langle L\rangle$ is nil and hence the maximality assumption implies that $N_0 = 0$. But then N is nilpotent so $N\langle L\rangle$ is nil and we conclude that $N = 0$.

For our last application we return to crossed products. Since the L-prime radical of R is "generated" by L-invariant nilpotent ideals, it trivially extends to a nil ideal of $R * U(L)$. In fact it is an immediate consequence of [5] that it extends to the prime radical of $R * U(L)$.

PROPOSITION 2.6. *Let $R * U(L)$ be given. If N is the L-prime radical of R, then $N * U(L)$ is the prime radical of $R * U(L)$.*

PROOF. Let P be a prime ideal of $S = R * U(L)$. Then $I = P \cap R$ is an L-prime ideal of R and $P \supseteq I * U(L)$. But the latter ideal is also prime by [5, Theorem 2.6]. Thus the intersection of all primes of S is equal to

$$\bigcap I * U(L) = \left(\bigcap I\right) * U(L)$$

where I runs through the L-prime ideals of R. Since $N = \bigcap I$, the result follows.

§3. The upper bound

In this final section we obtain upper bounds for $J(R * U(L))$ and we determine this Jacobson radical when R is right Noetherian or a p.i. algebra. We start by choosing a well-ordered K-basis $\{x_1, x_2, x_3, \ldots\}$ for L. By the Poincaré–Birkhoff–Witt theorem, $U(L)$ has a K-basis consisting of monomials μ of the form

$$\mu = \bar{x}_{i_1}\bar{x}_{i_2}\cdots\bar{x}_{i_n}$$

with $i_1 \leqq i_2 \leqq \cdots \leqq i_n$. These then also form an R-basis for $R * U(L)$. We order this basis of monomials first by degree and then lexicographically within monomials of the same degree. The following is [5, Lemma 2.4].

LEMMA 3.1. *Let $A \lhd R * U(L)$. For each monomial τ define A_τ to be the set of $r \in R$ such that there exists*

$$\alpha = \sum_{\mu} r_\mu \mu \in A$$

with $\deg \alpha \leqq \deg \tau$ *and* $r = r_\tau$. *Then A_τ is an ideal of R. Furthermore for each integer $n \geqq 0$*

$$A_n = \sum_{\deg \tau = n} A_\tau$$

is an L-invariant ideal of R.

The next result is the key ingredient in our argument. Since P is not assumed to be L-stable, $P(R * U(L))$ is only a right ideal of $R * U(L)$. Nevertheless with care we are able to consider products modulo this right ideal.

LEMMA 3.2. *Let $R * U(L)$ be given with $L \neq 0$. Assume that P is a prime ideal of R such that*

(i) *every nonzero ideal of R/P contains a regular element of R/P,*

(ii) *P contains no nonzero L-stable ideal of R.*

Then $J(R * U(L)) = 0$.

PROOF. Suppose by way of contradiction that $A = J(R * U(L)) \neq 0$. Then A contains a nonzero element of degree n for some n. Furthermore since $L \neq 0$, by suitably multiplying this element by a monomial, we may assume that $n > 0$. In the notation of the preceding lemma, A_n is a nonzero L-invariant ideal of R. Hence by asumption (ii), $A_n \subsetneqq P$ and thus for some τ of degree n, $A_\tau \subsetneqq P$. In other words there exists

$$\rho = \sum r_\mu \mu \in A$$

with $\deg \rho = n$ and $r_\tau \notin P$ for some monomial τ of degree n.

For this particular ρ, we may suppose that τ is maximal in the lexicographical order with $r_\tau \notin P$. In other words, if $\mu > \tau$ then $r_\mu \in P$. By (i), $Rr_\tau R$ contains a regular element modulo P. Say this element is $a = \Sigma_i u_i r_\tau v_i$ with $u_i, v_i \in R$. We now replace ρ by

$$\alpha = \sum_i u_i \rho v_i \in A.$$

Since $r_\mu \in P$ for $\mu > \tau$, it follows that if

$$\alpha = \sum a_\mu \mu$$

then $a_\tau = a$ is regular modulo P and that $a_\mu \in P$ for $\mu > \tau$. Furthermore $\deg \alpha = n = \deg \tau$.

Since $\alpha \in J(R * U(L))$, $1 + \alpha$ is invertible and we let $\beta = \Sigma_\lambda b_\lambda \lambda$ be its inverse. Thus

$$1 = \beta(1+\alpha) = \left(\sum_\lambda b_\lambda \lambda\right)\left(1 + \sum_\mu a_\mu \mu\right).$$

We show that all $b_\lambda \in P$. Suppose by way of contradiction that this is not the case and choose the monomial σ maximal in the support of β with $b_\sigma \notin P$. Say $\deg \sigma = m$ and let η be the monomial $\sigma\tau$ permuted into its natural order. In other words,

$$\sigma\tau = \eta + \text{lower degree terms.}$$

We compute the coefficient, with elements of R written on the left, of the monomial η in $\beta(1+\alpha) = 1$. Since $n > 0$

$$\deg \eta = \deg \sigma\tau = m + n > 0$$

so this coefficient is surely zero.

We now consider the contributions of the individual factors to this η-term. Suppose first that $\lambda > \sigma$. Then $b_\lambda \in P$ and hence all coefficients in $b_\lambda \lambda(1+\alpha)$ are contained in P. Thus we may assume that $\lambda \leqq \sigma$. Since $\deg \alpha = \deg \tau = n > 0$, the only other terms that have an η-contribution are of the form $b_\lambda \lambda a_\mu \mu$ with $\deg \lambda = m$ and $\deg \mu = n$. In this case, since

$$\lambda a_\mu = a_\mu \lambda + \text{lower degree terms}$$

we see that the degree $m + n$ contribution is $b_\lambda a_\mu \lambda\mu$ and of course $\lambda\mu$ is a monomial of degree $m + n$ plus lower degree terms. In particular, if $\mu > \tau$, then $a_\mu \in P$ and again we get an η-term in P. The remaining factors to consider have $\lambda \leqq \sigma$, $\mu \leqq \tau$ and

$$\lambda\mu = \sigma\tau + \text{lower degree terms.}$$

Thus we must have $\lambda = \sigma$, $\mu = \tau$ and this yields an η-term precisely equal to $b_\sigma a_\tau = b_\sigma a$. Combining all of these terms, using the fact that their sum is zero, we see that $b_\sigma a \in P$. But a is regular modulo P and $b_\sigma \notin P$ so we have a contradiction.

We have therefore shown that β is contained in the right ideal $P(R * U(L))$. Hence $1 = \beta(1 + \alpha)$ is also in this right ideal, a contradiction since $1 \notin P$. Thus $J(R * U(L)) = A = 0$.

We remark that the hypothesis of Lemma 3.2 implies that R is an L-prime ring with no nonzero L-invariant nil ideal. Indeed if A, B are L-stable ideals with $AB = 0$, then $AB \subseteq P$ so say $A \subseteq P$. Hypothesis (ii) then implies that $A = 0$. On the other hand if A is an L-stable nil ideal, then A can contain no regular element modulo P. Thus $A \subseteq P$ by (i) and again $A = 0$. We can now obtain an upper bound for $J(R * U(L))$.

THEOREM 3.3. *Let $R * U(L)$ be given with $L \neq 0$. Assume that R has a family of prime ideals P_j such that every nonzero ideal of R/P_j contains a regular element of the latter ring. Then*

$$J(R * U(L)) \subseteq N * U(L)$$

where N is the largest L-stable ideal contained in $\bigcap_j P_j$.

PROOF. For each j let I_j be the largest L-stable ideal of R contained in P_j. Then $\bigcap_j I_j$ is an L-stable ideal contained in $\bigcap_j P_j$. Thus by definition, $\bigcap_j I_j \subseteq N$.

Observe that $\bar{R}_j = R/I_j$ has a prime ideal $\bar{P}_j = P_j/I_j$ which satisfies (i) and (ii) of the preceding lemma. Indeed $\bar{R}_j/\bar{P}_j \cong R/P_j$ implies (i) and the definition of I_j yields (ii). We conclude from Lemma 3.2 that

$$R * U(L)/I_j * U(L) \simeq \bar{R}_j * U(L)$$

is semiprimitive.

Finally since $J(R * U(L))$ maps into the radical of every homomorphic image of $R * U(L)$, the above implies that

$$J(R * U(L)) \subseteq \bigcap_j I_j * U(L)$$

$$= \left(\bigcap_j I_j \right) * U(L) \subseteq N * U(L)$$

as required.

This of course has a number of consequences. However the content of the next result is really Corollary 2.4. In characteristic 0, minimal primes are always L-stable by Lemma 1.1(iv) and therefore the difficulties encountered in the proof of Lemma 3.2 disappear.

COROLLARY 3.4. *Let* $R * U(L)$ *be given with* $L \neq 0$ *and* K *a field of characteristic* 0. *Assume that for each minimal prime* P *of* R, *every nonzero ideal of* R/P *contains a regular element of the latter ring. Then* $J(R * U(L)) = N * U(L)$ *where* N *is the prime radical of* R. *Furthermore* $J(R * U(L))$ *is nil.*

PROOF. By Corollary 2.4, $N * U(L)$ is a nil ideal and hence contained in $J(R * U(L))$. In the other direction we use Theorem 3.3 and the fact that N is the intersection of the minimal primes of R.

More interesting is

COROLLARY 3.5. *Let* $R * U(L)$ *be a crossed product with* $L \neq 0$ *and assume that* R *is either right Noetherian, a p.i. algebra or a ring with no nilpotent elements. Then* $J(R * U(L)) = N * U(L)$ *where* N *is the largest* L*-invariant nil ideal of* R. *Furthermore* $J(R * U(L))$ *is nil.*

PROOF. By Corollary 2.5 for rings of the first two types, and trivially for the third, we have $N * U(L)$ nil and hence $N * U(L) \subseteq J(R * U(L))$.

In the other direction we use Theorem 3.3. If R is either right Noetherian or a p.i. algebra and if P is any prime ideal of R, then every nonzero ideal of R/P contains a regular element of the latter ring. Furthermore the intersection of all such primes is the prime radical and hence a nil ideal. We conclude from Theorem 3.3 that $J(R * U(L)) \subseteq N * U(L)$. Finally if R has no nilpotent elements then (see [9, Theorem 1.1.1]) R is a subdirect product of domains and, by Theorem 3.3 again, $J(R * U(L)) = 0 = N * U(L)$.

Finally we obtain information about $J(R * U(L))$ with no assumption on the primes of R. Instead we suppose that R has no nonzero nil ideals. Recall that the monomials in $R * U(L)$ are well ordered. If $\alpha = \Sigma\, a_\mu \mu \in R * U(L)$ and if σ is the largest monomial in its support, we say that σ is the monomial degree of α and that a_σ is its leading coefficient.

LEMMA 3.6. *Let $\alpha \in R * U(L)$ be invertible with* $\deg \alpha \geqq 1$. *If the leading coefficient a_σ commutes with α, then a_σ is nilpotent.*

PROOF. Write $a = a_\sigma$. Since α is invertible there exists β with $\alpha\beta = 1 = a^0$. Now choose γ of minimal support size such that $\alpha\gamma = a^n$ for some n. If $\gamma \neq 0$ let $c_\tau \tau$ be its leading term. Since $\deg \sigma = \deg \alpha \geqq 1$ and $a^n \in R$, we have $ac_\tau = a_\sigma c_\tau = 0$. Hence since $a\alpha = \alpha a$,

$$a^{n+1} = a(\alpha\gamma) = \alpha(a\gamma).$$

But $a\gamma$ has smaller support than γ, a contradiction. We conclude that $\gamma = 0$ and therefore that $a^n = 0$.

PROPOSITION 3.7. *Let $R * U(L)$ be given and assume that R has no nonzero nil ideals. If $J(R * U(L)) \neq 0$, then $R \cap J(R * U(L)) \neq 0$.*

PROOF. Assume that the smallest monomial degree of any nonzero element of $J(R * U(L))$ is σ. If A is the set of σ-coefficients of all elements $\alpha \in J(R * U(L))$ with mon-deg $\alpha \leqq \sigma$, then A is clearly a nonzero two-sided ideal of R. If $\sigma = 1$, then $R \cap J(R * U(L)) = A \neq 0$ as required.

Suppose by way of contradiction that $\sigma \neq 1$ so $\deg \sigma \geqq 1$. Let $0 \neq a \in A$ and let $\alpha \in J(R * U(L))$ with leading coefficient $a = a_\sigma$. Then $a\alpha - \alpha a \in J(R * U(L))$ has monomial degree smaller than σ so $a\alpha - \alpha a = 0$. Since $\deg \sigma \geqq 1$, $1 + \alpha$ also has leading coefficient $a = a_\sigma$ and $1 + \alpha$ is invertible. It follows from Lemma 3.6 that a is nilpotent. Thus A is a nonzero nil ideal of R, a contradiction.

The following is well known so we only sketch its proof.

LEMMA 3.8. *Let L' be a Lie subalgebra of L. Then*

$$J(R * U(L)) \cap R * U(L') \subseteq J(R * U(L')).$$

PROOF. By extending a well ordered K-basis of L' to one of L we deduce from the Poincaré–Birkhoff–Witt theorem that

$$R * U(L) = R * U(L') \oplus C$$

where C is a complementary left $R * U(L')$-module. This then implies that if

$\alpha \in R * U(L')$ is invertible in $R * U(L)$, then it is invertible in $R * U(L')$. With this we see that $J(R * U(L)) \cap R * U(L')$ is a quasi-regular ideal of $R * U(L')$ and hence contained in its Jacobson radical.

We can now prove

THEOREM 3.9. *Let $R * U(L)$ be given and assume that R has no nonzero nil ideal. If either*

(i) *JR contains no nonzero L-stable ideal, or*

(ii) *some $0 \neq x \in L$ acts as an inner derivation on R,*

then $J(R * U(L)) = 0$.

PROOF. Assume by way of contradiction that $J(R * U(L)) \neq 0$. Then by Proposition 3.7, $A = R \cap J(R * U(L)) \neq 0$ and certainly A is an L-stable ideal of R. By Lemma 3.8 with $L' = 0$ we have $A \subseteq JR$. Hence if (i) is satisfied we have an appropriate contradiction.

Assume (ii) holds and let $0 \neq x \in L$ act like the inner derivation induced by $b \in R$ and set $L' = Kx$. Then by Lemma 3.8,

$$A \subseteq J(R * U(L)) \cap R * U(L') \subseteq J(R * U(L')).$$

Let $a \in A$. Then $a(\bar{x} - b) \in J(R * U(L'))$ so $1 + a(\bar{x} - b)$ is invertible in $R * U(L')$. But $\bar{x} - b$ acts trivially on R so a commutes with $1 + a(\bar{x} - b)$. It follows from Lemma 3.6 that a is nilpotent and hence that A is nil, a contradiction.

COROLLARY 3.10. *Let $R * U(L)$ be given with* char $K = 0$. *Then*

$$J(R * U(L)) \subseteq M * U(L)$$

where M is the largest L-invariant ideal in JR. Furthermore if some $0 \neq x \in L$ acts as an inner derivation on R, then

$$J(R * U(L)) \subseteq N * U(L)$$

where N is the nil radical of R.

PROOF. For the first part it suffices to show that

$$R * U(L)/M * U(L) = (R/M) * U(L)$$

is semiprimitive. Hence since $M \subseteq JR$ we can assume, replacing R by R/M, that JR contains no nonzero L-stable ideal. But the nil radical is L-stable, by

Lemma 1.1(i), so we conclude that R has no nonzero nil ideals and Theorem 3.9(i) yields the result.

For the second part we note that N is L-stable, by Lemma 1.1, and therefore it suffices to show that

$$R * U(L)/N * U(L) = (R/N) * U(L)$$

is semiprimitive. But R/N has no nonzero nil ideals and some $0 \neq x \in L$ acts as an inner derivation on R/N so, this time, Theorem 3.9(ii) yields the result.

It is quite possible that the second part of the above holds with only the assumption that $\operatorname{char} K = 0$ and $L \neq 0$. In view of the proof of Theorem 3.9(ii), this problem reduces to showing that $J(R[x;\delta]) \cap R$ is nil for any Ore extension $R[x;\delta]$. The equations involved in a potential proof of this fact seem to be quite complicated. We close with one such which is interesting but certainly not readily useful.

LEMMA 3.11. *Let $a \in J(R[x;\delta]) \cap R$ and define a_i inductively by $a_1 = a$ and $a_{i+1} = (1+\delta(a_i))^{-1}a_i$. Then $a_1a_2\cdots a_n = 0$ for some $n \geqq 1$.*

PROOF. Note first that $A = J(R[x;\delta]) \cap R$ is a δ-stable ideal of R contained in JR, by Lemma 3.8. With this we see that the sequence $a_1, a_2, a_3, \ldots$ in A is well defined. Indeed, by induction, since $a_i \in A$ we have $\delta(a_i) \in A \subseteq JR$ so $(1+\delta(a_i))^{-1} \in R$ exists and $a_{i+1} = (1+\delta(a_i))^{-1}a_i \in A$. For convenience set $c_n = a_1a_2\cdots a_{n-1}$ with $c_1 = 1$. Since $a \in J(R[x;\delta])$, $1 + ax$ is invertible and there exists β with $\beta(1+ax) = 1$ or, in other words, $\beta(1+a_1x) = c_1$.

We can now choose β of minimal degree with $\beta(1+a_nx) = c_n$ for some $n \geqq 1$. If $\beta \neq 0$, let b be its leading coefficient so that clearly $ba_n = 0$. Now

$$\begin{aligned} c_{n+1} = c_na_n = \beta(1+a_nx)a_n &= \beta a_n(1+xa_n) \\ &= \beta a_n(1+\delta(a_n)+a_nx) \\ &= \beta a_n(1+\delta(a_n))\cdot(1+a_{n+1}x). \end{aligned}$$

But $ba_n = 0$ implies that $\deg \beta a_n(1+\delta(a_n)) < \deg \beta$, a contradiction. Thus $\beta = 0$ and $c_n = 0$.

Notice that each a_i and hence each $a_1a_2\cdots a_n$ is a product of a's with units interspersed. Thus if R is commutative we obtain the known result that $J(R[x;\delta]) \cap R$ is nil.

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